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On a Method of Newman and a Theorem of Bernstein

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INTRODUCTION

D. J. Newman, in [2], derives precise upper bounds for uniform rational approximations to e^x on $[-1, 1]$. He writes

$$e^x = e^{z/2} e^{\bar{z}/2} \sim R_{n,m}(z/2)\, R_{n,m}(\bar{z}/2),$$

where

$$x = \tfrac{1}{2}(z + \bar{z}) \qquad \text{and} \qquad |z| = 1$$

and where $R_{n,m}$ is the (n, m) Padé approximate to e^z. The critical observation is that the approximant $R_{n,m}(z/2)\, R_{n,m}(\bar{z}/2)$ is a rational function of type (n, m) in the variable $x = (\frac{1}{2})(z + \bar{z})$. (See also Szabados [4].) It is our intention to further examine this approach.

Let E_ρ, $\rho > 1$, be the closed ellipse in the complex plane with foci at ± 1 and with semiaxes $\frac{1}{2}(\rho \pm \rho^{-1})$. Suppose that f is analytic and non-zero on a neighbourhood of E_ρ. As in [2, p. 25], for

$$z = x + iy \qquad \text{and} \qquad x^2 + y^2 = 1,$$

we have

$$f(x) = F(z)\, F(\bar{z}),$$

where

$$\log\left(f\left(\frac{z + z^{-1}}{2}\right)\right) = \log F(z) + \log F(\bar{z}).$$

Furthermore, we may assume that $F(z)$ is analytic on $D_\rho = \{z : |z| \leqslant \rho\}$.

Let Π_n denote the set of algebraic polynomials of degree at most n. We

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say that a rational function $p(z)/q(z)$ is of type (n, m) if $p \in \Pi_n$ and $q \in \Pi_m$. The normal (n, m) Padé approximant to a function g analytic in a neighbourhood of zero is the (n, m) rational function p_n/q_m if it exists, that satisfies

$$g(z)\, q_m(z) - p_n(z) = z^{m+n+1} h(z),$$

where h is analytic in a neighbourhood of zero and where $q_m(0) \neq 0$.

The following theorem generalizes a well-known result of S. N. Bernstein [1, p. 76] and reduces to his result in the polynomial case. Let $\|\cdot\|_I$ denote the supremum norm on the set I.

THEOREM. *Suppose that f is analytic and non-zero in a neighbourhood of E_ρ. Let F be defined as above and suppose that R is the normal (n, m) Padé approximant to F. If*

$$\|F(z) - R(z)\|_{D_\rho} \leqslant A \qquad \text{and} \qquad \|F(z)\|_{D_\rho} \leqslant B \tag{$*$}$$

then

$$\|f(x) - S(x)\|_{[-1,1]} \leqslant \frac{3A(A+B)\rho}{\rho^{n+m}(\rho-1)^2},$$

where $S(x) = R(z/2)\, R(\bar{z}/2)$ is a rational function of type (n, m).

Proof. If $|z| = 1$ then

$$\begin{aligned}
&F(z)\, F(\bar{z}) - R(z)\, R(\bar{z}) \\
&\quad = F(z)\, F(1/z) - R(z)\, R(1/z) \\
&\quad = \frac{z^{n+m+1}[F(z) - R(z)]\, F(1/z)}{z^{n+m+1}} + \frac{z^{-(n+m+1)}[F(1/z) - R(1/z)]\, R(z)}{z^{-(n+m+1)}} \\
&\quad = \frac{z^{n+m+1}}{2\pi i} \int_{\alpha_1} \frac{(F(\zeta) - R(\zeta))\, F(1/\zeta)\, d\zeta}{\zeta^{n+m+1}(\zeta - z)} \\
&\qquad + \frac{z^{n+m+1}}{2\pi i} \int_{\alpha_2} \frac{(F(\zeta) - R(\zeta))\, F(1/\zeta)\, d\zeta}{\zeta^{n+m+1}(\zeta - z)} \\
&\qquad + \frac{z^{-(n+m+1)}}{2\pi i} \int_{\alpha_1} \frac{(F(1/\zeta) - R(1/\zeta))\, R(\zeta)\, d\zeta}{\zeta^{-(n+m+1)}(\zeta - z)} \\
&\qquad + \frac{z^{-(n+m+1)}}{2\pi i} \int_{\alpha_2} \frac{(F(1/\zeta) - R(1/\zeta))\, R(\zeta)\, d\zeta}{\zeta^{-(n+m+1)}(\zeta - z)} \\
&\quad = I_1(z) + I_2(z) + I_3(z) + I_4(z),
\end{aligned}$$

where α_1 is the circle of radius ρ taken counter-clockwise and α_2 is the circle of radius $1/\rho$ taken clockwise. It is easily verified from the definitions that

$$\frac{F(z)-R(z)}{z^{n+m+1}}$$

is analytic on the annulus $\{z\colon 1/\rho \leqslant |z| \leqslant \rho\}$ and hence, that the preceding application of Cauchy's integral formula is valid. For $|z|=1$,

$$|I_1(z)| \leqslant \frac{1}{2\pi}\int_{\alpha_1} \frac{AB\,d\xi}{\rho^{n+m+1}(\rho-1)} \leqslant \frac{AB}{\rho^{n+m}(\rho-1)}.$$

Similarly, for $|z|=1$,

$$|I_4(z)| \leqslant \frac{A(A+B)}{\rho^{n+m+1}(\rho-1)}.$$

We now estimate $I_2(z)$ and $I_3(z)$. First, for $|w|<\rho$,

$$F(w)-R(w) = \frac{w^{n+m+1}}{2\pi i}\int_{\alpha_1} \frac{(F(\zeta)-R(\zeta))\,d\zeta}{\zeta^{n+m+1}(\zeta-w)}$$

and for $w \in \alpha_2$,

$$|F(w)-R(w)| \leqslant \frac{A}{\rho^{2n+2m+1}(\rho-\rho^{-1})}.$$

Thus, for $|z|=1$,

$$|I_2(z)| \leqslant \frac{AB}{\rho^{n+m}(\rho-\rho^{-1})(\rho-1)}$$

and

$$|I_3(z)| \leqslant \frac{A(A+B)\rho}{\rho^{n+m}(\rho-\rho^{-1})(\rho-1)}.$$

Combining the above estimates yields, for $|z|=1$,

$$|F(z)\,F(\bar{z}) - R(z)\,R(z)| \leqslant \frac{3A(A+B)}{\rho^{n+m-1}(\rho-1)^2}$$

whence the result follows.

Condition (*) of the above theorem is always satisfied in the polynomial case. In the general case Szabados [4] shows that there exists a function f analytic in E_ρ so that

$$\lim_{n\to\infty} \sup(R_{n,n}(f))^{1/n} = 1/\rho,$$

where

$$R_{n,n}(f) = \inf_{p_n, q_n \in \pi_n} \| p_n/q_n - f \|_{[-1,1]}.$$

Thus, we cannot hope to omit assumption (*) completely from the theorem.

APPROXIMATING $(x-\rho)^{1/2}$

Assume that p/q is the normal (n, n) Padé approximant at the point $z = 1$ to the function $z^{1/2}$. Then,

$$p(z) - q(z)\sqrt{z} = (1-z)^{2n+1} h(z)$$

and

$$p(z^2) - q(z^2)z = (1-z)^{2n+1}(1+z)^{2n+1} h(z^2),$$

where $h(z)$ is analytic on $C - (-\infty, 0]$. Since $p(z^2) - q(z^2)z$ is a polynomial of degree $2n+1$ with $2n+1$ roots at 1, it follows that

$$p(z) - q(z)\sqrt{z} = (1-\sqrt{z})^{2n+1}, \tag{1}$$

where q is suitably normalized. Expanding (1) and comparing coefficients yields

$$\begin{aligned} q(z) &= \sum_{k=0}^{n} \binom{2n+1}{2k+1} z^k \\ &= \frac{1}{2\sqrt{z}} \left[(1+\sqrt{z})^{2n+1} - (1-\sqrt{z})^{2n+1}\right] \end{aligned} \tag{2}$$

and

$$p(z) = \tfrac{1}{2}\left[(1+\sqrt{z})^{2n+1} + (1-\sqrt{z})^{2n+1}\right]. \tag{3}$$

(That we may assume the existence of the normal (n, n) Padé approximant is now clear from (1), (2) and (3).) From (2) one deduces that q has only real-negative roots and that for $|z-1| \leqslant 1$,

$$\left|\frac{p(z)}{q(z)}-\sqrt{z}\right| \leqslant \frac{|(1-\sqrt{z})^{2n+1}|}{q_n(0)} \leqslant \frac{1}{2n+1}.$$

If $p_\rho(z)/q_\rho(z)$ is the normal (n, n) Padé approximant to $(\rho - z)^{1/2}$ then

$$\|p_\rho(z)/q_\rho(z)-(\rho-z)^{1/2}\|_{D_\rho} \leqslant \frac{\rho^{1/2}}{2n+1}.$$

Thus, by the Theorem, there exists $S_{n,n}$ a rational function of type (n, n) so that

$$\left\| S_{n,n}(x)-\frac{(\rho^2+1-2\rho x)^{1/2}}{(2\rho)^{1/2}}\right\|_{[-1,1]} \leqslant \frac{3\rho^2}{n\rho^{2n}(\rho-1)^2}.$$

If we set $\alpha = (\rho^2+1)/2\rho$ we get

$$\|S_{n,n}(x)-(\alpha-x)^{1/2}\|_{[-1,1]} \leqslant \left(\frac{12\alpha^2}{n(\alpha^2-1)}\right) \cdot \frac{1}{(\alpha+\sqrt{\alpha^2-1})^{2n}}.$$

These types of rational approximations to $(\alpha - x)^{1/2}$ converge at least as fast as $(\alpha+\sqrt{\alpha^2-1})^{-2n}$ while polynomial approximations only behave like $(\alpha+\sqrt{\alpha^2-1})^{-n}$. See [3 p. 437] for a more general discussion of the derivation of Padé approximations to $x^{1/2}$ and related functions.

References

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